

Variational formulation of time-fractional parabolic equations *

Michael Karkulik†

Abstract

We consider initial/boundary value problems for time-fractional parabolic PDE of order $0 < \alpha < 1$ with Caputo fractional derivative (also called fractional diffusion equations in the literature). We prove well-posedness of corresponding variational formulations based entirely on fractional Sobolev-Bochner spaces, and clarify the question of possible choices of the initial value.

Key words: Fractional diffusion equation, Initial value/boundary value problem, Well-posedness

AMS Subject Classification: 26A33, 35K15, 35R11

1 Introduction

Physical phenomena based on standard diffusion, where the mean square displacement of a diffusing particle scales linearly with time $\langle x(t)^2 \rangle \sim t$, are typically modeled by partial differential equations involving standard (i.e., integer order) differential operators. So-called *anomalous diffusion*, on the other hand, is characterized by non-linear scaling. For example, a diverse number of systems exhibit anomalous diffusion which follows the power-law $\langle x(t)^2 \rangle \sim t^\alpha$ with $0 < \alpha < 1$ (subdiffusion) or $1 < \alpha < 2$ (superdiffusion). Systems with such power-laws include ones with constrained pathways such as fractal, disordered, or porous media, polymers, aquifers, and quantum systems, among others. We refer to [18] for an extensive overview on the subject. In the latter work, the authors list various ways how to model anomalous diffusion processes. For problems involving external fields or boundary conditions, the most natural way is to consider partial differential equations involving so-called *fractional differential operators*. In the work at hand, we consider a time-fractional parabolic initial/boundary value problem of the form

$$\begin{aligned} \partial_t^\alpha u - \Delta u &= f && \text{in } (0, T) \times \Omega, \\ u &= 0 && \text{on } (0, T) \times \partial\Omega, \\ u &= g && \text{for } \{0\} \times \Omega, \end{aligned} \tag{1}$$

*Supported by Conicyt Chile through project FONDECYT 1170672.

†Departamento de Matemática, Universidad Técnica Federico Santa María, Avenida España 1680, Valparaíso, Chile, mkarkulik.mat.utfsm.cl, email: michael.karkulik@usm.cl

where $(0, T)$ is a time interval and $\Omega \subset \mathbb{R}^n$ a spatial Lipschitz domain. Here, $-\Delta$ is the spatial Laplacian, $1/2 < \alpha < 1$, and ∂_t^α is a fractional time derivative of order α . More specifically, we will use the so-called Caputo derivative, which is defined formally by

$$\partial_t^\alpha u(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} u'(s) ds.$$

Recently, researchers have started to analyze finite element methods with respect to their ability to approximate solutions of fractional differential equations. While this started with classical Galerkin finite element methods for steady-state fractional diffusion equations as in [9, 12], numerical methods for time-dependent fractional partial differential equations include time-stepping methods [8, 13, 14], Discontinuous Galerkin methods [19, 20], as well as methods based on the Laplace transform [17]. It goes without saying that this list is far from being exhaustive. We mention here also the numerical approach from [21] which is based on the extension theory by Caffarelli and Silvestre [4]. The aforementioned numerical methods are usually based on a variational formulation of the equation under consideration. Existing works on variational formulations of time-fractional parabolic partial differential equations are scarce; as to our knowledge, the works [27, 24, 1] are of relevance in connection with our model problem (1) (for Semigroup theory for related Volterra integral equations see [23]). These works have in common that (i) their functional analytic setting is not based exclusively on classical Sobolev regularity in time, but rather involves the operator ∂_t^α , and that (ii) the initial value g is taken from $L_2(\Omega)$. The goal of the present work is to derive the well-posedness of variational formulations set up in classical Sobolev-Bochner spaces and to clarify the question of regularity needed for the initial data. Now, as our functional analytic setting is based only on Sobolev regularity, a result of this kind is specifically interesting for numerical analysis of the equation (1). Indeed, approximation results for functions with certain Sobolev regularity are well known and ubiquitous in numerical analysis. The property (i) is owed to the fact that there is no rigorous definition of time-fractional derivatives on fractional Sobolev-Bochner spaces available. It sure is true that operators defined between real valued Sobolev spaces $L_2(J) \rightarrow L_2(J)$ do extend to vector-valued counterparts $L_2(J; X) \rightarrow L_2(J; X)$ (for X a Hilbert space, this is a classical result of Marcinkiewicz and Zygmund [16]), but the fact that we are dealing with Sobolev regularity $H^\alpha(J; X)$ in time needs some care and additional analysis. To that end, we will show first that the fractional Caputo derivative is a linear and bounded operator on a time-fractional Sobolev-Bochner space. This way, we can consider a variational formulation of (1) based exclusively on Sobolev regularity, which resembles classical variational formulations for parabolic equations. Regarding the point (ii), the choice of $g \in L_2(\Omega)$ as initial value is indeed admissible, but one has to bear in mind the following: While the space $L_2(J; \tilde{H}^1(\Omega)) \cap L_2(J; H^{-1}(\Omega))$, used in variational formulations of parabolic equations, is continuously embedded in $C(\bar{J}; L_2(\Omega))$, this is no longer true for the equation (1). We will show that the space of solutions of our variational formulation of (1) is continuously embedded in $C(\bar{J}; H^{1-1/\alpha-\varepsilon}(\Omega))$ for all $\varepsilon > 0$. Our main result is then well-posedness of the variational formulation, cf. Theorem 2.

2 Mathematical setting and main results

2.1 Sobolev and Bochner spaces

We denote by $\Omega \subset \mathbb{R}^d$ a (spatial) Lipschitz domain, and by $J = (0, T)$ for $T > 0$ a temporal interval. We use Lebesgue and Sobolev spaces $L_2(\Omega)$ and $\tilde{H}^1(\Omega)$, the tilde denoting vanishing trace on the boundary $\partial\Omega$. The fractional Sobolev spaces $\tilde{H}^s(\Omega)$ for $s \in (0, 1)$ are defined by the K-method of interpolation as $\tilde{H}^s(\Omega) := [L_2(\Omega), \tilde{H}^1(\Omega)]_{s,2}$, cf. [26], where

$$[B_0, B_1]_{s,2} := \left\{ u \in B_0 \mid \|u\|_{[B_0, B_1]_{s,2}} < \infty \right\}, \quad \|u\|_{[B_0, B_1]_{s,2}}^2 := \int_0^\infty t^{-2s-1} K_{[B_0, B_1]}(t, u)^2 dt$$

with the K-functional

$$K_{[B_0, B_1]}(t, u)^2 := \inf_{w \in B_1} \|u - w\|_{B_0}^2 + t^2 \|w\|_{B_1}^2.$$

The topological dual of a Banach space X is denoted by X' , and we define $H^{-s}(\Omega) := \tilde{H}^s(\Omega)'$ as the topological duals with respect to the extended $L_2(\Omega)$ scalar product (\cdot, \cdot) , and duality will be denoted by $\langle \cdot, \cdot \rangle$. We set $\tilde{H}^0(\Omega) := L_2(\Omega)$. In time, we additionally use Lebesgue and Sobolev spaces $L_2(J)$ and $H^\alpha(J)$, for $\alpha \in (0, 1]$. The $L_2(J)$ scalar product will also be denoted by (\cdot, \cdot) , but it will always be clear which scalar product we are using. We use the notation Du to denote the distributional derivative of a function u given on J . For $\alpha \in (0, 1)$ the norm on the space $H^\alpha(J)$ is given by

$$\|f\|_{H^\alpha(J)}^2 := \|f\|_{L_2(J)}^2 + |f|_{H^\alpha(J)}^2 < \infty, \quad \text{where } |f|_{H^\alpha(J)}^2 := \int_J \int_J \frac{|f(s) - f(t)|^2}{|s - t|^{2\alpha+1}} ds dt.$$

We mention that $H^\alpha(J) = [L_2(J), H^1(J)]_{\alpha,2}$ with equivalent norms. We set $H^{-s}(J) := \tilde{H}^s(J)'$ and $\tilde{H}^{-s}(J) := H^s(J)$ for $s \in (0, 1)$. For a Banach space X , we use the Bochner space $L_2(J; X)$ of functions $f : J \rightarrow X$ which are strongly measurable with respect to the Lebesgue measure ds on J and

$$\|f\|_{L_2(J; X)}^2 := \int_J \|f(s)\|_X^2 ds < \infty.$$

For w a measurable, positive function on J , we denote by $L_2(J, w; X)$ the w -weighted Lebesgue space of strongly measurable functions with norm

$$\|f\|_{L_2(J, w; X)}^2 := \int_J w(s)^2 \|f(s)\|_X^2 ds < \infty.$$

We say that a function $f \in L_2(J; X)$ has a weak time-derivative $\partial_t f \in L_2(J; X)$, if

$$\int_J \partial_t f \varphi = - \int_J f \partial_t \varphi \quad \text{for all } \varphi \in C_0^\infty(J). \quad (2)$$

Note that this last integral has to be understood in the sense of Bochner. We define the space $H^1(J; X)$ as the space of functions with

$$\|f\|_{H^1(J; X)}^2 := \|f\|_{L_2(J; X)}^2 + \|\partial_t f\|_{L_2(J; X)}^2.$$

For $0 < \alpha < 1$, we also use the fractional Sobolev-Bochner space $H^\alpha(J; X)$ of ds -strongly measurable functions $f : J \rightarrow X$ with

$$\|f\|_{H^\alpha(J; X)}^2 := \|f\|_{L_2(J; X)}^2 + |f|_{H^\alpha(J; X)}^2 < \infty, \quad \text{where } |f|_{H^\alpha(J; X)}^2 := \int_J \int_J \frac{\|f(s) - f(t)\|_X^2}{|s - t|^{2\alpha+1}} ds dt.$$

We will also use these Bochner spaces on \mathbb{R} instead of J . For a recent introduction to Bochner spaces, we refer to [11].

2.2 Fractional time derivative on Bochner spaces

For $0 < \beta < 1$, we define the left and right-sided Riemann-Liouville fractional integral operators

$${}_0D^{-\beta}u(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} u(s) ds \quad \text{and} \quad D_T^{-\beta}u(t) := \frac{1}{\Gamma(\beta)} \int_t^T (s-t)^{\beta-1} u(s) ds,$$

where Γ is Euler's Gamma function. For sufficiently smooth functions u , the left-sided Caputo fractional derivative ∂^α for $\alpha \in (0, 1)$ is defined as $\partial^\alpha u := {}_0D^{\alpha-1}Du$. We will show below in Lemma 10 that the tensorised version ${}_0D^{-\beta} \otimes I$ defined by $({}_0D^{-\beta} \otimes I)(u \otimes x) := ({}_0D^{-\beta}u) \otimes x$ can be extended uniquely to a linear and bounded operator ${}_0D^{-\beta} : L_2(J; X) \rightarrow H^\beta(J; X)$ for a Hilbert space X . This allows us to prove the following result. The proof will be carried out below in Section 3.3.

Theorem 1. *Let ∂_t be the weak time derivative defined in (2). Then, for $\alpha \in (1/2, 1)$, the operator $\partial_t^\alpha := {}_0D^{\alpha-1} \circ \partial_t$ is linear and bounded as $\partial_t^\alpha : H^\alpha(J; H^{-1}(\Omega)) \rightarrow L_2(J; H^{-1}(\Omega))$. \square*

2.3 Variational formulation and main result

Our variational formulation of (1) is the following: Given $f \in L_2(J; H^{-1}(\Omega))$ and $g \in H^{1-1/\alpha+\delta}(\Omega)$ for some $\delta > 0$, find $u \in L_2(J; \tilde{H}^1(\Omega))$ with $u \in H^\alpha(J; H^{-1}(\Omega))$ such that

$$\langle \partial_t^\alpha u, v \rangle + (\nabla u, \nabla v) = (f, v) \quad \text{for all } v \in \tilde{H}^1(\Omega), \quad (3)$$

almost everywhere in J , and $u(0, \cdot) = g(\cdot)$. The duality $\langle \partial_t^\alpha u, v \rangle$ in (3) makes sense due to the mapping properties of ∂_t^α from Theorem 1, and the initial condition makes sense as we will show in Corollary 9 below that $L_2(J; \tilde{H}^1(\Omega)) \cap H^\alpha(J; H^{-1}(\Omega))$ is continuously embedded in $C(\bar{J}; H^{1-1/\alpha-\varepsilon}(\Omega))$ for all $\varepsilon > 0$. The following theorem is our main result and will be proven below in Section 3.3.

Theorem 2. *The variational formulation (3) is well posed: there exists a unique solution u , and*

$$\|u\|_{L_2(J; \tilde{H}^1(\Omega))} + \|u\|_{H^\alpha(J; H^{-1}(\Omega))} \leq C_\delta \left(\|g\|_{H^{1-1/\alpha+\delta}(\Omega)} + \|f\|_{L_2(J; H^{-1}(\Omega))} \right).$$

The constant $C_\delta > 0$ depends only on δ . □

Remark 3. It is textbook knowledge that there holds the continuous embedding

$$L_2(J; \tilde{H}^1(\Omega)) \cap H^1(J; H^{-1}(\Omega)) \hookrightarrow C(\overline{J}; L_2(\Omega)).$$

In the present case, we have the embedding

$$L_2(J; \tilde{H}^1(\Omega)) \cap H^\alpha(J; H^{-1}(\Omega)) \hookrightarrow C(\overline{J}; H^{1-1/\alpha-\varepsilon}(\Omega))$$

for all $\varepsilon > 0$, cf. Lemma 8 below. The reason for the missing power of ε is that we use the embedding result $H^{1/2+\varepsilon}(J; X) \hookrightarrow C(\overline{J}; X)$. Furthermore, note that the stability estimate of Theorem 2 involves the $H^{1-1/\alpha+\delta}(\Omega)$ norm of the initial data g , and the constant C_δ is expected to blow up for $\delta \rightarrow 0$.

3 Technical results

3.1 Fractional integral and differential operators

We have the following results. The first point is an extension of a recent result in [12] and can be found in [7, Lemma 5], while the second point is part of the proof of [7, Lemma 6]. The third part can be found in [15, Lem. 2.7].

Lemma 4. (i) *For every $s \in \mathbb{R}$ with $-\beta \leq s$ and $\beta > 0$, the operators ${}_0D^{-\beta}$ and $D_T^{-\beta}$ can be extended to bounded linear operators $\tilde{H}^s(J) \rightarrow H^{s+\beta}(J)$.*

(ii) *For $0 < \beta < 1$, the operator ${}_0D^{-\beta}$ is elliptic on $H^{-\beta/2}(J)$.*

(iii) *For $0 < \beta < 1$ and $u, v \in L_2(J)$ it holds $(D_0^{-\beta}u, v) = (u, D_T^{-\beta}v)$.* □

The Mittag-Leffler function arises naturally in the study of fractional differential equations. We refer to [6, Section 18.1] for an overview. It is defined as

$$E_{n_1, n_2}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(kn_1 + n_2)}.$$

According to [22, Thm. 1.6], for $z \in \mathbb{R}$,

$$E_{n_1, n_2}(z) \lesssim \frac{1}{1 + |z|}, \tag{4}$$

and due to [5, Thm. 4.3],

$${}_0D^{\alpha-1}DE_{\alpha,1}(\lambda t^\alpha) = \lambda E_{\alpha,1}(\lambda t^\alpha). \quad (5)$$

Furthermore, by [25], $E_{\alpha,1}(-z)$ is completely monotone for $0 < \alpha \leq 1$ and positive z , in particular,

$$E'_{\alpha,1}(-z) \geq 0 \quad \text{for positive } z. \quad (6)$$

We will need the following result on fractional seminorms, which combines the H^s norm and the dual norm of the distributional derivative.

Lemma 5. *Let $s \in (0, 1)$ be fixed. There holds*

$$|u|_{H^s(J)} \lesssim \|Du\|_{H^{s-1}(J)} \quad \text{for all } u \in H^s(J),$$

where Du is the distributional derivative of u .

Proof. As $u \in L_2(J)$, it holds $Du \in H^{-1}(J)$. We can write $u = D\psi + c$ with $c \in \mathbb{R}$, where $\psi \in \tilde{H}^1(J)$ is the unique solution of $(D\psi, D\varphi) = (u, D\varphi)$ for all $\varphi \in \tilde{H}^1(J)$. Then, $\|\psi\|_{\tilde{H}^1(J)} \lesssim \|u\|_{L_2(J)}$, and due to the definition of the distributional derivative we see

$$|(u, D\psi)| = |(Du, \psi)| \lesssim \|Du\|_{H^{-1}(J)} \|\psi\|_{\tilde{H}^1(J)} \lesssim \|Du\|_{H^{-1}(J)} \|u\|_{L_2(J)}.$$

We conclude that for $u \in L_2(J)$, it holds

$$\|u\|_{L_2(J)}^2 = (u, D\psi) + (u, c) \lesssim \|Du\|_{H^{-1}(J)} \|u\|_{L_2(J)} + (u, c).$$

Now we apply this estimate to $u - \bar{u}$, where \bar{u} denotes the mean value of u , and obtain

$$\|u - \bar{u}\|_{L_2(J)} \lesssim \|Du\|_{H^{-1}(J)}. \quad (7)$$

The standard Poincaré inequality states that

$$\|u - \bar{u}\|_{H^1(J)} \lesssim \|Du\|_{L_2(J)}. \quad (8)$$

The $H^s(J)$ norm can equivalently be obtained by the K-method of interpolation via

$$\|u - \bar{u}\|_{H^s(J)}^2 \simeq \|u - \bar{u}\|_{[L_2(J), H^1(J)]_{s,2}}^2 = \int_0^\infty t^{-2s} \left(\inf_{v \in H^1(J)} \|u - \bar{u} - v\|_{L_2(J)}^2 + t^2 \|v\|_{H^1(J)}^2 \right) \frac{dt}{t}.$$

Using (7) and (8), we obtain

$$\begin{aligned} \inf_{v \in H^1(J)} \|u - \bar{u} - v\|_{L_2(J)}^2 + t^2 \|v\|_{H^1(J)}^2 &\leq \inf_{\substack{v \in H^1(J) \\ \bar{v}=0}} \|u - \bar{u} - v\|_{L_2(J)}^2 + t^2 \|v\|_{H^1(J)}^2 \\ &\lesssim \inf_{\substack{v \in H^1(J) \\ \bar{v}=0}} \|Du - Dv\|_{H^{-1}(J)}^2 + t^2 \|Dv\|_{L_2(J)}^2 \end{aligned}$$

Next we use that for $w \in L_2(J)$ there is a $\psi \in H^1(J)$ with $\bar{\psi} = 0$ such that $D\psi = w$. We conclude

$$\|u - \bar{u}\|_{H^s(J)}^2 \lesssim \int_0^\infty t^{-2s} \left(\inf_{w \in L_2(J)} \|Du - w\|_{H^{-1}(J)}^2 + t^2 \|w\|_{L_2(J)}^2 \right) \frac{dt}{t}.$$

By definition, the right-hand side is $\|Du\|_{[H^{-1}(J), L_2(J)]_{s,2}}^2$, which is equivalent to $\|Du\|_{H^{s-1}(J)}^2$. This concludes the proof. \square

The next lemma establishes a norm equivalence on a fractional Sobolev space.

Lemma 6. *Let $1/2 < s < 1$. Then, for all $u \in H^s(J)$,*

$$|u|_{H^s(J)} \sim \|{}_0D^{s-1}Du\|_{L_2(J)}.$$

Proof. We have

$$\|{}_0D^{s-1}Du\|_{L_2(J)} \lesssim \|Du\|_{\tilde{H}^{s-1}(J)} \lesssim \|Du\|_{H^{s-1}(J)} \lesssim \|u\|_{H^s(J)}.$$

Here, the first estimate follows from Lemma 4, and the second one can be found in [10, Lem. 5]. To see the third estimate, recall that D is the distributional derivative, and hence $\|Du\|_{H^{-1}(J)} \leq \|u\|_{L_2(J)}$ as well as $\|Du\|_{L_2(J)} \leq \|u\|_{H^1(J)}$. The third estimate now follows from an interpolation argument. The fact that $Du = D(u - \bar{u})$ for the mean value \bar{u} of u and Poincaré's inequality show

$$|u|_{H^s(J)} \gtrsim \|{}_0D^{s-1}Du\|_{L_2(J)}.$$

To show the converse estimate, we take $u \in C^\infty(\bar{J})$ and estimate with Lemmas 5 and 4

$$\begin{aligned} |u|_{H^s(J)}^2 &\lesssim \|Du\|_{H^{s-1}(J)}^2 \lesssim ({}_0D^{2(s-1)}Du, Du) \\ &= ({}_0D^{s-1}Du, D_T^{s-1}Du) \leq \|{}_0D^{s-1}Du\|_{L_2(J)} \|D_T^{s-1}Du\|_{L_2(J)}. \end{aligned}$$

Here, the identity follows from Lemma 4, (iii). Due to Lemma 4, it also holds $\|D_T^{s-1}Du\|_{L_2(J)} \lesssim \|Du\|_{\tilde{H}^{s-1}(J)} \lesssim \|u\|_{H^s(J)}$, where the second estimate was already shown at the beginning of this proof. Applying the whole argument to $u - \bar{u}$ and using Poincaré's inequality finally shows the statement. \square

3.2 Sobolev and Bochner spaces

For $s \in (-1, 0]$ we have the interpolation estimate

$$\|u\|_{H^s(\Omega)} \leq C(s) \|u\|_{H^{-1}(\Omega)}^{(1-s)/2} \cdot \|u\|_{\tilde{H}^1(\Omega)}^{(1+s)/2}, \quad (9)$$

with $C(s) > 0$ a constant depending only on s . This estimate follows for $s = 0$ by duality, and for $s \in (-1, 0)$ using additionally [26, 1.3.3 (g)] and the fact that duality and interpolation commute,

cf. [26, 1.11.2]. For a measurable set $M \subset \mathbb{R}$, we denote by $\mathbf{1}_M$ the characteristic function on M , and for a function $\phi : J \rightarrow \mathbb{R}$ and $x \in X$, we define $\phi \otimes x := J \rightarrow X$ as $(\phi \otimes x)(s) := \phi(s)x$. We denote by $\mathcal{L}(J)$ the σ -algebra of all ds -measurable sets on J , and by $S(J)$ the set of simple functions. It is known that the following subsets are dense for J bounded or $J = \mathbb{R}$,

$$S(J; X) := \left\{ \sum_{i=1}^n \mathbf{1}_{A_i} \otimes x_i \mid A_i \in \mathcal{L}(J), x_i \in X \text{ for all } i = 1, \dots, n, n \in \mathbb{N} \right\} \subset L_2(J; X),$$

$$S^\infty(J; X) := \left\{ \sum_{i=1}^n \varphi_i \otimes x_i \mid \varphi_i \in C_c^\infty(\mathbb{R}), x_i \in X \text{ for all } i = 1, \dots, n, n \in \mathbb{N} \right\} \subset H^1(J; X).$$

We assume from now on that the Banach spaces X are reflexive; this implies that they have the so-called Radon-Nikodým property, cf. [11, Thm. 1.95], which is sufficient and necessary in order to have that $L_2(J; X)'$ is isometrically isomorphic to $L_2(J; X')$, cf. [11, Thm. 1.84]. We can extend ∂_t for $u \in L_2(J; X)$ by defining $\partial_t u \in H_0^1(J; X')'$ as

$$- \int_J \langle u(s), \partial_t \varphi(s) \rangle ds \quad \text{for } \varphi \in H_0^1(J; X').$$

Then, we have that $\partial_t : L_2(J; X) \rightarrow H_0^1(J; X')'$ is bounded. Furthermore, $\partial_t : H^1(J; X) \rightarrow L_2(J; X) = L_2(J; X')'$ is bounded, and by interpolation, we have that for $s \in (0, 1)$

$$\partial_t : [L_2(J; X), H^1(J; X)]_s \rightarrow [H_0^1(J; X')', L_2(J; X')']_s \quad (10)$$

is bounded. We will need the following results on interpolation of Sobolev-Bochner spaces.

Lemma 7. *There holds*

$$[L_2(\mathbb{R}; X), H^1(\mathbb{R}; X)]_s = H^s(\mathbb{R}; X)$$

and

$$[\tilde{H}^1(J; X')', L_2(J; X')']_s = [\tilde{H}^1(J; X'), L_2(J; X')]_s' = [L_2(J; X'), \tilde{H}^1(J; X')]_{1-s}' = \tilde{H}^{1-s}(J; X')'.$$

Proof. The first identity is due to [11, Thm. 2.91], and the second and third identities are well-known results in interpolation theory, cf. [26, 1.11.2]. The last identity is a variant of the first one with bounded interval and zero traces. Using extension theorems, its proof can in fact be reduced to the first identity. In the case of scalar-valued Sobolev spaces, we refer to [3, Thm. 14.2.3] for details. \square

Next, we will establish continuous embeddings for the function space of our variational formulation.

Lemma 8. *Suppose that $\alpha \in (0, 1)$, $s \in (-1, 0]$ and $0 < r$ are such that $r < \alpha(1 - s)/2$. Then, we have the continuous embedding*

$$L_2(J; \tilde{H}^1(\Omega)) \cap H^\alpha(J; H^{-1}(\Omega)) \hookrightarrow H^r(J; H^s(\Omega)).$$

Proof. It is clear that $\|u\|_{L_2(J;H^s(\Omega))} \leq \|u\|_{L_2(J;\tilde{H}^1(\Omega))}$. To bound the H^r -seminorm, we write for $r < \alpha(1-s)/2$

$$2r+1 = (2\alpha+1)\frac{1-s}{2} + (1-\varepsilon)\frac{1+s}{2}$$

for some $\varepsilon > 0$. The interpolation estimate (9) and the inequalities of Cauchy-Schwarz and Young then yield

$$\begin{aligned} \int_J \int_J \frac{\|u(s) - u(t)\|_{\tilde{H}^s(\Omega)}^2}{|s-t|^{2r+1}} dt ds &\lesssim \int_J \int_J \frac{\|u(s) - u(t)\|_{H^{-1}(\Omega)}^{1-s} \|u(s) - u(t)\|_{\tilde{H}^1(\Omega)}^{1+s}}{|s-t|^{2r+1}} dt ds \\ &\lesssim \int_J \left(\int_J \frac{\|u(s) - u(t)\|_{H^{-1}(\Omega)}^2}{|s-t|^{2\alpha+1}} dt \right)^{(1-s)/2} \\ &\quad \left(\int_J \frac{\|u(s) - u(t)\|_{\tilde{H}^1(\Omega)}^2}{|s-t|^{1-\varepsilon}} dt \right)^{(1+s)/2} ds \\ &\lesssim \int_J \int_J \frac{\|u(s) - u(t)\|_{H^{-1}(\Omega)}^2}{|s-t|^{2\alpha+1}} dt ds + \int_J \int_J \frac{\|u(s) - u(t)\|_{\tilde{H}^1(\Omega)}^2}{|s-t|^{1-\varepsilon}} dt ds \end{aligned}$$

and as $\varepsilon > 0$, the last integral can be bounded by $\|u\|_{L_2(J;\tilde{H}^1(\Omega))}$. \square

Corollary 9. *Suppose that $\alpha \in (0, 1)$ and $s \in (-1, 0]$ are such that $s < 1 - 1/\alpha$. Then, we have the continuous embedding*

$$L_2(J; \tilde{H}^1(\Omega)) \cap H^\alpha(J; H^{-1}(\Omega)) \hookrightarrow C(\bar{J}; H^s(\Omega)).$$

Proof. If $s < 1 - 1/\alpha$, then $1/2 < \alpha(1-s)/2$, and according to Lemma 8 there holds the continuous embedding $L_2(J; \tilde{H}^1(\Omega)) \cap H^\alpha(J; H^{-1}(\Omega)) \hookrightarrow H^{1/2+\varepsilon}(J; H^s(\Omega))$ for a sufficiently small $\varepsilon > 0$. According to [11, Thm. 2.95] there also holds the continuous embedding $H^{1/2+\varepsilon}(J; H^s(\Omega)) \hookrightarrow C(\bar{J}; H^s(\Omega))$, and this proves the statement. \square

The next lemma shows that the Riemann-Liouville fractional integral operators can be extended in the canonical way (i.e., by tensorisation) to Sobolev-Bochner spaces.

Lemma 10. *Suppose that X is a Hilbert space and $0 < \beta < 1/2$. Then, the operator*

$${}_0D^{-\beta} \otimes I := \begin{cases} S(J; X) \rightarrow H^\beta(J; X) \\ \sum_{i=1}^n x_i \mathbf{1}_{A_i} \mapsto \sum_{i=1}^n ({}_0D^{-\beta} \mathbf{1}_{A_i}) x_i \end{cases}$$

can be extended uniquely to a linear and bounded operator ${}_0D^{-\beta} : L_2(J; X) \rightarrow H^\beta(J; X)$. The same statement is true for the operator $D_T^{-\beta} \otimes I$.

Proof. It follows from Lemma 4 (i) that the operator ${}_0D^{-\beta} : L_2(J) \rightarrow L_2(J)$ is bounded. Furthermore, it is a positive operator, i.e., ${}_0D^{-\beta}(\mathbf{1}_{A_i}) \geq 0$ on J . It is then easy to see, cf. [11, Thm. 2.3], that

$$\|{}_0D^{-\beta} \otimes Iu\|_{L_2(J;X)} \leq \|{}_0D^{-\beta}\|_{L_2(J) \rightarrow L_2(J)} \|u\|_{L_2(J;X)} \quad \text{for } u \in S(J;X), \quad (11)$$

and as $S(J;X)$ is dense in $L_2(J;X)$, we obtain boundedness ${}_0D^{-\beta} : L_2(J;X) \rightarrow L_2(J;X)$. Next, we will follow the ideas developed in [12, Thm. 3.1]. Denoting by $\tilde{f} \in L_2(\mathbb{R})$ the extension of f by zero, it holds ${}_0D^{-\beta}f(x) = {}_{-\infty}D^{-\beta}\tilde{f}(x)$. Denote by $\mathcal{F} : L_2(\mathbb{R}) \rightarrow L_2(\mathbb{R})$ the Fourier transformation. Then, the operator $\mathcal{F} \otimes I$ extends to an isometry $\mathcal{F} : L_2(\mathbb{R};X) \rightarrow L_2(\mathbb{R};X)$: For general operators, this is a classical result by Marcinkiewicz and Zygmund [16], cf. [11, Thm. 2.9], but in the present case of the Fourier transformation it can be seen readily by using density of simple functions $S(\mathbb{R};X)$ in $L_2(\mathbb{R};X)$ and the Plancherel theorem for the scalar-valued Fourier transformation. Furthermore, for $u \in L_2(\mathbb{R};X)$ we have $\mathcal{F}\mathcal{F}u = \mathcal{P}u$ with $\mathcal{P}u(x) = u(-x)$ the parity operator. For a function $\varphi = \sum_{i=1}^n \varphi_i \otimes x_i \in S^\infty(\mathbb{R};X)$, we conclude

$$\begin{aligned} \|\varphi\|_{H^1(\mathbb{R};X)}^2 &= \left\| \sum_{i=1}^n \varphi_i \otimes x_i \right\|_{L_2(\mathbb{R};X)}^2 + \left\| \sum_{i=1}^n \partial_t \varphi_i \otimes x_i \right\|_{L_2(\mathbb{R};X)}^2 \\ &= \left\| \sum_{i=1}^n \mathcal{F}\varphi_i \otimes x_i \right\|_{L_2(\mathbb{R};X)}^2 + \left\| \sum_{i=1}^n \mathcal{F}(\partial_t \varphi_i) \otimes x_i \right\|_{L_2(\mathbb{R};X)}^2 \\ &= \|g\mathcal{F}\varphi\|_{L_2(\mathbb{R};X)}^2, \end{aligned}$$

with weight function $g(\omega) := \sqrt{1+\omega^2}$. By density, this shows that $\mathcal{F} \otimes I$ can be extended to an isometry $\mathcal{F} : H^1(\mathbb{R};X) \rightarrow L_2(\mathbb{R};g;X)$. By interpolation and Lemma 7, we conclude that $\mathcal{F} : H^s(\mathbb{R};X) \rightarrow L_2(\mathbb{R};g^s;X)$ is bounded. To show that this operator is an isometry, consider a decomposition $\tilde{u}_0 + \tilde{u}_1 = \mathcal{F}u$ with $\tilde{u}_0 \in L_2(\mathbb{R};X)$, $\tilde{u}_1 \in L_2(\mathbb{R};g;X)$. From $\|\mathcal{F}\tilde{u}_1\|_{H^1(\mathbb{R};X)} = \|\mathcal{F}\mathcal{F}\tilde{u}_1\|_{L_2(\mathbb{R};g;X)} = \|\tilde{u}_1\|_{L_2(\mathbb{R};g;X)}$ we conclude $\mathcal{F}\tilde{u}_1 \in H^1(\mathbb{R};X)$ and due to $\mathcal{F}\tilde{u}_0 + \mathcal{F}\tilde{u}_1 = \mathcal{F}\mathcal{F}u = \mathcal{P}u$ we have that $\mathcal{P}\mathcal{F}\tilde{u}_0 + \mathcal{P}\mathcal{F}\tilde{u}_1 = u$ is a decomposition of u . Hence,

$$\begin{aligned} \|\tilde{u}_0\|_{L_2(\mathbb{R};X)}^2 + t\|\tilde{u}_1\|_{L_2(\mathbb{R};g;X)}^2 &= \|\mathcal{F}\tilde{u}_0\|_{L_2(\mathbb{R};X)}^2 + t\|\mathcal{F}\tilde{u}_1\|_{H^1(\mathbb{R};X)}^2 \\ &= \|\mathcal{P}\mathcal{F}\tilde{u}_0\|_{L_2(\mathbb{R};X)}^2 + t\|\mathcal{P}\mathcal{F}\tilde{u}_1\|_{H^1(\mathbb{R};X)}^2, \end{aligned}$$

which implies $K_{[L_2(\mathbb{R};X), H^1(\mathbb{R};X)]}(t, u)^2 \leq K_{[L_2(\mathbb{R};X), L_2(\mathbb{R};g;X)]}(t, \mathcal{F}u)$. This shows that $\mathcal{F} : H^s(\mathbb{R};X) \rightarrow L_2(\mathbb{R};g^s;X)$ is an isometry. Next, for a simple function $u \in S(\mathbb{R};X)$,

$$\begin{aligned} \|{}_{-\infty}D^{-\beta} \otimes Iu\|_{H^s(\mathbb{R};X)}^2 &= \int_{\mathbb{R}} g(\omega)^{2s} \|\mathcal{F}{}_{-\infty}D^\beta u(\omega)\|_X^2 d\omega \\ &\lesssim \int_{|\omega| \leq 1} \|\mathcal{F}{}_{-\infty}D^\beta u(\omega)\|_X^2 d\omega + \int_{|\omega| > 1} (\omega^{-2} + 1)^s \|\mathcal{F}u(\omega)\|_X^2 d\omega \\ &\leq \int_{\mathbb{R}} \|\mathcal{F}{}_{-\infty}D^\beta u(\omega)\|_X^2 d\omega + \int_{|\omega| > 1} \|\mathcal{F}u(\omega)\|_X^2 d\omega \\ &\lesssim \int_{\mathbb{R}} \|{}_{-\infty}D^\beta u(s)\|_X^2 ds + \int_{\mathbb{R}} \|u(s)\|_X^2 ds \lesssim \|u\|_{L_2(\mathbb{R};X)}^2, \end{aligned}$$

and by density we get the desired result. The proof for $D_T^{-\beta}$ follows along the same lines. \square

Lemma 11. *The operator ${}_0D^{-\beta}u \otimes I$ has a unique extension as bounded and linear operator*

$${}_0D^{-\beta} \otimes I : H^\beta(J; \tilde{H}^1(\Omega))' \rightarrow L_2(J; H^{-1}(\Omega)).$$

Proof. For $u = \sum_{i=1}^n \mathbf{1}_{A_i} \otimes u_i \in S(J; H^{-1}(\Omega))$, $v = \sum_{i=1}^n \mathbf{1}_{A_i} \otimes v_i \in S(J; \tilde{H}^1(\Omega))$, we compute

$$\begin{aligned} (u, D_T^{-\beta} \otimes Iv) &= \int_J \langle u(s), D_T^{-\beta} \otimes Iv(s) \rangle ds \\ &= \sum_{i,j=1}^n \langle u_i, v_i \rangle \int_J \mathbf{1}_{A_i}(s) D_T^{-\beta} \mathbf{1}_{A_j}(s) ds \\ &= \sum_{i,j=1}^n \langle u_i, v_i \rangle \int_J {}_0D^{-\beta} \mathbf{1}_{A_i}(s) \mathbf{1}_{A_j}(s) ds \\ &= ({}_0D^{-\beta} \otimes Iu, v). \end{aligned} \tag{12}$$

As $H^\beta(J; \tilde{H}^1(\Omega))$ is dense in $L_2(J; \tilde{H}^1(\Omega))$, $L_2(J; H^{-1}(\Omega)) = L_2(J; \tilde{H}^1(\Omega))'$ is dense in $H^\beta(J; \tilde{H}^1(\Omega))'$. According to Lemma 10, $D_T^{-\beta} : L_2(J; \tilde{H}^1(\Omega)) \rightarrow H^\beta(J; \tilde{H}^1(\Omega))$ is bounded, and hence the equality (12) shows that ${}_0D^{-\beta}$ can be extended as stipulated. This finishes the proof. \square

3.3 Proof of the main theorems

Proof of Theorem 1. Using the boundedness (10) of ∂_t and Lemmas 7 and 11, we conclude that

$${}_0D^{\alpha-1} \circ \partial_t : H^\alpha(J; H^{-1}(\Omega)) \rightarrow \tilde{H}^{1-\alpha}(J; \tilde{H}^1(\Omega))' = H^{1-\alpha}(J; \tilde{H}^1(\Omega))' \rightarrow L_2(J; H^{-1}(\Omega))$$

is bounded. \square

Proof of Theorem 2. We mimic the proof for parabolic PDE. Take $(w_k)_{k \geq 1}$ the $L_2(\Omega)$ -orthonormal basis of eigenfunctions and $(\lambda_k)_{k \geq 1}$ the eigenvalues of $-\Delta$. We make the ansatz

$$u_m(t) := \sum_{k=1}^m d_m^k(t) w_k.$$

Now, we are looking for $d_m^k : J \rightarrow \mathbb{R}$ such that

$$\begin{aligned} \partial_t^\alpha d_m^k(t) + \lambda_k d_m^k(t) &= \langle f(t), w_k \rangle, \quad k = 1, \dots, m, \\ d_m^k(0) &= \langle g, w_k \rangle, \quad k = 1, \dots, m. \end{aligned} \tag{13}$$

According to [2, Thm. 2.1], cf. [5, Thm. 7.2] and [15, Chapter 3.1], the solutions to these equations are given uniquely by

$$d_m^k(t) = \langle g, w_k \rangle E_\alpha(-\lambda_k t^\alpha) + \psi_k(t), \quad k = 1, \dots, m,$$

where $\psi_k(t) := \alpha \int_0^t \langle f(t-s), w_k \rangle s^{\alpha-1} E'_\alpha(-\lambda_k s^\alpha) ds$. In order to obtain energy estimates for the u_m , we can extend the calculations carried out in [24]. However, as we aim at weaker initial values, we need a finer analysis. First, using the bound (4), we have for $\varepsilon \in [0, 1]$

$$|E_{\alpha,1}(z)| \lesssim \frac{1}{1+|z|} \leq |z|^{-(1-\varepsilon)}.$$

Furthermore, $\alpha \int_J t^{\alpha-1} E'_{\alpha,1}(-\lambda_k t^\alpha) dt = \lambda_k^{-1}(1 - E_{\alpha,1}(-\lambda_k T^\alpha))$, and $t^{\alpha-1} E'_{\alpha,1}(-\lambda_k t^\alpha) \geq 0$ due to (6). Hence, we see

$$\int_J |t^{\alpha-1} E'_{\alpha,1}(-\lambda_k t^\alpha)| dt \lesssim \lambda_k^{-1}. \quad (14)$$

We conclude that for $2\alpha(1-\varepsilon) < 1$, it holds

$$\|E_{\alpha,1}(-\lambda_k(\cdot)^\alpha)\|_{L_2(J)}^2 \leq C_\varepsilon \lambda_k^{-2(1-\varepsilon)}. \quad (15)$$

Furthermore, due to Lemma 6, the identity (5) and the previous estimate, we also have

$$|E_\alpha(-\lambda_k(\cdot)^\alpha)|_{H^\alpha(J)}^2 \sim \lambda_k^2 \|E_{\alpha,1}(-\lambda_k(\cdot)^\alpha)\|_{L_2(J)}^2 \leq C_\varepsilon \lambda_k^{2\varepsilon}. \quad (16)$$

By Young's inequality and (14),

$$\|\psi_k\|_{L_2(J)}^2 \lesssim \left(\int_J \langle f(t), w_k \rangle^2 dt \right) \cdot \left(\int_J |t^{\alpha-1} E'_\alpha(-\lambda_k t^\alpha)| dt \right)^2 \lesssim \lambda_k^{-2} \int_J \langle f(t), w_k \rangle^2 dt. \quad (17)$$

According to [22, pp. 140], it holds $\partial_t^\alpha \psi_k(t) = -\langle f(t), w_k \rangle - \lambda_k \psi_k(t)$. Hence, using Lemma 6 and (17), we also see

$$|\psi|_{H^\alpha(J)}^2 \lesssim \int_J \langle f(t), w_k \rangle^2 dt. \quad (18)$$

Now choose $\varepsilon := (2-1/\alpha+\delta)/2$ and observe that $2\alpha(1-\varepsilon) = 1-\alpha\delta < 1$ and $-1+2\varepsilon = 1-1/\alpha+\delta$. Using (15) and (17), we estimate

$$\begin{aligned} \|u_m\|_{L_2(J; \tilde{H}^1(\Omega))}^2 &= \sum_{k=0}^m \lambda_k \|d_m^k\|_{L_2(J)}^2 \lesssim \sum_{k=0}^m \lambda_k^{-1+2\varepsilon} \langle g, w_k \rangle^2 + \sum_{k=0}^m \lambda_k^{-1} \int_J \langle f(t), w_k \rangle^2 dt \\ &\lesssim \|g\|_{H^{1-1/\alpha+\delta}(\Omega)}^2 + \|f\|_{L_2(J; H^{-1}(\Omega))}^2. \end{aligned}$$

Using (16) and (18), we can analogously estimate

$$\|u_m\|_{H^\alpha(J; H^{-1}(\Omega))}^2 \lesssim \|g\|_{H^{1-1/\alpha+\delta}(\Omega)}^2 + \|f\|_{L_2(J; H^{-1}(\Omega))}^2.$$

Therefore, $(u_m)_{m \in \mathbb{N}}$ is a bounded sequence in $L_2(J; \tilde{H}^1(\Omega))$ and in $H^\alpha(J; H^{-1}(\Omega))$, and we conclude that there is a subsequence $(u_{m_k})_{k \in \mathbb{N}}$ which converges weakly to some $u \in L_2(J; \tilde{H}^1(\Omega))$

and to some $\tilde{u} \in H^\alpha(J; H^{-1}(\Omega))$. It follows that u also converges weakly in $L_2(J; H^{-1}(\Omega))$ to u as well as to \tilde{u} , which yields $u = \tilde{u}$. Taking into account the construction of the u_m and invoking the weak limit, we obtain for all $v \in L_2(J; \tilde{H}^1(\Omega))$

$$\int_J \langle \partial_t^\alpha u, v \rangle + \langle \nabla u, \nabla v \rangle dt = \int_J \langle f, v \rangle dt.$$

Note that due to Corollary 9, u_{m_k} also converges weakly to u in $C([0, T]; H^{1-1/\alpha+\delta}(\Omega))$, hence $g = u_{m_k}(0) \rightarrow u(0)$. This yields $u(0) = g$, and we conclude that u is a weak solution. As for uniqueness, if u is a weak solution with vanishing data, then the functions $u_k(t) := (u(t), w_k)$ solve the equations (13) with vanishing right-hand side, and hence $u_k(t) = 0$. \square

Acknowledgement: The author would like to thank Vincent J. Ervin for his valuable comments.

References

- [1] M. Allen, L. Caffarelli, and A. Vasseur. A parabolic problem with a fractional time derivative. *Arch. Ration. Mech. Anal.*, 221(2):603–630, 2016.
- [2] J. H. Barrett. Differential equations of non-integer order. *Canadian J. Math.*, 6:529–541, 1954.
- [3] S. C. Brenner and L. R. Scott. *The mathematical theory of finite element methods*, volume 15 of *Texts in Applied Mathematics*. Springer, New York, third edition, 2008.
- [4] L. Caffarelli and L. Silvestre. An extension problem related to the fractional Laplacian. *Comm. Partial Differential Equations*, 32(7-9):1245–1260, 2007.
- [5] K. Diethelm. *The analysis of fractional differential equations*, volume 2004 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2010. An application-oriented exposition using differential operators of Caputo type.
- [6] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. a. Tricomi. *Higher transcendental functions. Vol. III*. Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1955 original.
- [7] V. Ervin, T. Führer, N. Heuer, and M. Karkulik. DPG method with optimal test functions for a fractional advection diffusion equation. *J. Sci. Comput.*, 2017. Accepted for publication.
- [8] V. J. Ervin, N. Heuer, and J. P. Roop. Numerical approximation of a time dependent, nonlinear, space-fractional diffusion equation. *SIAM J. Numer. Anal.*, 45(2):572–591, 2007.
- [9] V. J. Ervin and J. P. Roop. Variational formulation for the stationary fractional advection dispersion equation. *Numer. Methods Partial Differential Equations*, 22(3):558–576, 2006.

- [10] N. Heuer. Additive Schwarz method for the p -version of the boundary element method for the single layer potential operator on a plane screen. *Numer. Math.*, 88(3):485–511, 2001.
- [11] T. Hytönen, J. van Neerven, M. Veraar, and L. Weis. *Analysis in Banach Spaces*. Springer International Publishing, 2016. Volume I: Martingales and Littlewood-Paley Theory.
- [12] B. Jin, R. Lazarov, J. Pasciak, and W. Rundell. Variational formulation of problems involving fractional order differential operators. *Math. Comp.*, 84(296):2665–2700, 2015.
- [13] B. Jin, R. Lazarov, and Z. Zhou. Error estimates for a semidiscrete finite element method for fractional order parabolic equations. *SIAM J. Numer. Anal.*, 51(1):445–466, 2013.
- [14] B. Jin, R. Lazarov, and Z. Zhou. Two fully discrete schemes for fractional diffusion and diffusion-wave equations with nonsmooth data. *SIAM J. Sci. Comput.*, 38(1):A146–A170, 2016.
- [15] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and applications of fractional differential equations*, volume 204 of *North-Holland Mathematics Studies*. Elsevier Science B.V., Amsterdam, 2006.
- [16] J. Marcinkiewicz and A. Zygmund. Quelques inégalités pour les opérations linéaires. *Fundamenta Mathematicae*, 32(1):115–121, 1939.
- [17] W. McLean, I. H. Sloan, and V. Thomée. Time discretization via Laplace transformation of an integro-differential equation of parabolic type. *Numer. Math.*, 102(3):497–522, 2006.
- [18] R. Metzler and J. Klafter. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.*, 339(1):77, 2000.
- [19] K. Mustapha. Time-stepping discontinuous Galerkin methods for fractional diffusion problems. *Numer. Math.*, 130(3):497–516, 2015.
- [20] K. Mustapha, M. Nour, and B. Cockburn. Convergence and superconvergence analyses of HDG methods for time fractional diffusion problems. *Adv. Comput. Math.*, 42(2):377–393, 2016.
- [21] R. H. Nochetto, E. Otárola, and A. J. Salgado. A PDE approach to space-time fractional parabolic problems. *SIAM J. Numer. Anal.*, 54(2):848–873, 2016.
- [22] I. Podlubny. *Fractional differential equations*, volume 198 of *Mathematics in Science and Engineering*. Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.
- [23] J. Prüss. *Evolutionary integral equations and applications*. Modern Birkhäuser Classics. Birkhäuser/Springer Basel AG, Basel, 1993. [2012] reprint of the 1993 edition.

- [24] K. Sakamoto and M. Yamamoto. Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.*, 382(1):426–447, 2011.
- [25] W. R. Schneider. Completely monotone generalized Mittag-Leffler functions. *Exposition. Math.*, 14(1):3–16, 1996.
- [26] H. Triebel. *Interpolation theory, function spaces, differential operators*. Johann Ambrosius Barth, Heidelberg, second edition, 1995.
- [27] R. Zacher. Weak solutions of abstract evolutionary integro-differential equations in Hilbert spaces. *Funkcial. Ekvac.*, 52(1):1–18, 2009.